

ANALYTIC SUBSETS OF HILBERT SPACES [†]

VLADIMIR PESTOV

ABSTRACT. We show that every complete metric space is homeomorphic to the locus of zeros of an entire analytic map from a complex Hilbert space to a complex Banach space. As a corollary, every separable complete metric space is homeomorphic to the locus of zeros of an entire analytic map between two complex Hilbert spaces.

§1. Douady had observed [8] that every compact metric space is homeomorphic to the locus of zeros of an analytic (in fact, continuous polynomial) map between two suitable Banach spaces. In other words, every compact metric space is homeomorphic to an algebraic subset of a complex Banach space. Proving Douady's conjecture, the present author has shown [13] that every complete metric space is isometric to an algebraic subset of a complex Banach space. Hilbert spaces provide a particularly favourable setting for Banach analytic geometry (cf. [16]), and the following question suggested by Norm Dancer is very natural: what can be said about the topology of analytic subsets of Hilbert (or just reflexive Banach) spaces?

A previously known result belongs to Ramis [15] who embedded the Cantor set topologically in a Hilbert space as an algebraic subset.

Main Theorem. *Every complete metric space is homeomorphic to the precise locus of zeros of an entire analytic map from a complex Hilbert space to a complex Banach space.*

Call a Banach analytic space [7] an *Hilbert analytic space* if it is modelled on analytic subsets of an Hilbert space (zeros of Banach-valued analytic maps). The following corollary betters both Douady's and the present author's earlier results.

Corollary 1. *A paracompact topological space admits the structure of an Hilbert analytic space if and only if it is metrizable with a complete metric.*

Corollary 2. *Every separable complete metric space is homeomorphic to the locus of zeros of an entire map between two separable complex Hilbert spaces.*

Every closed subset of a separable real Hilbert space is the precise locus of zeros of a C^∞ functional (see e.g. [9], 2.C), but obviously this result does not extend to real analytic functionals. However, one has:

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Corollary 3. *Every separable complete metric space is homeomorphic to the locus of zeros of an entire functional on a separable real Hilbert space.*

The latter result indicates that theory of finitely defined analytic subsets of Banach (even Hilbert) manifolds [14], [15] only has topological substance in a complex case.

Remark 1. One can hardly expect the above Main Theorem to admit an isometric version. Already the unit segment $I = [0, 1]$ with the usual Euclidean distance cannot be isometrically embedded as an *analytic* subset in a Hilbert space. If $I \hookrightarrow \mathcal{H}$ is such an isometry, then I forms a segment of a *metric line* in \mathcal{H} and therefore a segment of a real affine line (cf. §3 in [4]). As an easy consequence, the locus of zeros of an appropriate non-constant real analytic map on an interval in \mathbb{R} contains an open non-void subinterval of I , which is impossible.

§2. All Banach spaces and algebras below are complex unless otherwise stated. In particular, the Sobolev spaces $W^{n,m}(\Omega)$ [1], the spaces $C^n[a, b]$, and the Banach algebra $\text{Lip}[0, 1]$ of Lipschitz functions are formed by complex-valued functions. We denote the locus of zeros of a mapping f by $V(f)$. An l_p -sum of a family of Banach spaces is denoted by \oplus^{l_p} . The *diagonal product* $\Delta_\alpha f_\alpha$ of a family of maps $f_\alpha: X \rightarrow Y_\alpha$ sends an $x \in X$ to $(f_\alpha(x)) \in \prod_\alpha Y_\alpha$.

Theorem 1 (Douady [8]; [15], 1.3.A). *The maximal ideal space, X , of a unital Banach algebra A is the locus of zeros of a continuous 2-polynomial on the Banach dual A' . \square*

A Sobolev space $W^{2,2}(0, 1)$ forms a separable Hilbert space and also a commutative unital Banach algebra under the pointwise multiplication ([1], 3.5 and 5.23).

Assertion 1. *The maximal ideal space of $W^{2,2}(0, 1)$ is canonically homeomorphic to the unit segment $[0, 1]$ in both the weak topology and the strong dual topology.*

Proof. The canonical embedding $C^2[0, 1] \rightarrow C^1[0, 1]$ is the composition of an obvious continuous embedding $i_2: C^2[0, 1] \rightarrow W^{2,2}(0, 1)$ and a continuous embedding $i_1: W^{2,2}(0, 1) \rightarrow C^1[0, 1]$ asserted by the Sobolev Embedding Theorem ([1], 5.4.I.C), as can be seen from [1], 5.2. Therefore, both i_1 and i_2 are homomorphisms of unital Banach algebras. Since the elements of maximal ideal spaces of both $C^1[0, 1]$ and $C^2[0, 1]$ are in a natural one-to-one correspondence with the points of $[0, 1]$ ([11], 1.3.3), the same is true for $W^{2,2}(0, 1)$. The Gelfand space of $\text{Lip}[0, 1]$ is canonically homeomorphic to $[0, 1]$ in both the strong and the weak topologies [15], [8], [13]. It follows that the dual operator to the composition of i_1 and the canonical homomorphism $C^1[0, 1] \rightarrow \text{Lip}[0, 1]$ maps the maximal ideal space of $\text{Lip}[0, 1]$ onto that of $W^{2,2}(0, 1)$ in a one-to-one fashion and continuously with respect to both pairs of topologies. \square

Assertion 2. *A separable Hilbert space \mathcal{H} contains a topological copy of the unit interval, I , as the locus of zeros of a continuous 2-polynomial on \mathcal{H} in such a way that the left endpoint $0_I \in I$ is the zero element of \mathcal{H} , and every closed subset of I is the intersection of I with the locus of zeros of a continuous 1-polynomial.*

Proof. Denote by \mathcal{H} the dual space of distributions $W^{2,2}(0, 1)'$. In view of the above two results, the translation, $X - 0_I$, of the maximal ideal space, $X \cong I$, of $W^{2,2}(0, 1)$ is the precise locus of zeros of a continuous 2-polynomial on \mathcal{H} . Every

closed subset, F , of $I \cong X$ is the locus of zeros of a suitable C^∞ -function, f , on I (e.g. [9], 2.C). Viewed as an element of $C^2[0, 1] \subset W^{2,2}(0, 1) \cong \mathcal{H}'$, this function determines a continuous linear functional \bar{f} on \mathcal{H} with the property $F = X \cap V(\bar{f})$. Finally, $F - 0_I = (X - 0_I) \cap V(\bar{f} - \bar{f}(0_I))$. \square

§3. Lemma 1. *The intersection of a family of subsets of a Banach space E , each of which is the locus of zeros of a continuous (resp. continuous homogeneous) polynomial map of degree $\leq n$ (resp. n) is the locus of zeros of a continuous (resp. continuous homogeneous) polynomial map of degree $\leq n$ (resp. n).*

Proof. Let $p_\alpha: E \rightarrow F_\alpha$ be continuous Banach space valued n -polynomial maps. One can assume without loss in generality that the norm [5], [7] of every homogeneous component of each polynomial p_α is ≤ 1 . The map $p \equiv \Delta_\alpha p_\alpha: E \rightarrow \oplus^{l_\infty} F_\alpha$ is a well-defined continuous polynomial of degree $\leq n$. Indeed, p can be represented as the sum of diagonal products $p^{(i)} \equiv \Delta_\alpha p_\alpha^{(i)}$ of homogeneous components of $p_\alpha = \sum_{i=0}^n p_\alpha^{(i)}$. Since for each $t \in \mathbb{C}$ one has $p^{(i)}(tx) = t^i p^{(i)}(x)$, each $p^{(i)}$ is a homogeneous polynomial of degree i ([5], coroll. 3.H). Furthermore, the norm of every homogeneous component of p does not exceed 1, and therefore every $p^{(i)}$ is continuous ([5], Th. 1; [7], prop. 1), as well as p itself. It is clear that $\cap_\alpha V(p_\alpha) = V(p)$. Finally, if each p_α is homogeneous of degree n , then so is p . \square

Lemma 2. *Let \mathcal{H}_α be a family of Hilbert spaces. Then the union $\cup_\alpha \mathcal{H}_\alpha$ of these spaces canonically embedded into their Hilbert direct sum $\mathcal{H} = \oplus^{l_2} \mathcal{H}_\alpha$ forms the locus of zeros of a continuous homogeneous polynomial map on \mathcal{H} of degree 2.*

Proof. For each pair of indices α, β , $\alpha \neq \beta$, let $\pi_{\alpha, \beta}$ be the projection of H to $H_\alpha \oplus H_\beta$, and let $i_{\alpha, \beta}$ stand for the canonical embedding of $H_\alpha \times H_\beta$ into $H_\alpha \hat{\otimes} H_\beta$ (the projective tensor product). The locus of zeros of the continuous homogeneous 2-polynomial $i_{\alpha, \beta} \circ \pi_{\alpha, \beta}$ consists of such elements $(x_\gamma) \in H$ that at most one of the two coordinates x_α, x_β is non-vanishing. The intersection of sets $V(i_{\alpha, \beta} \circ \pi_{\alpha, \beta})$ over all pairs (α, β) consists of all $(x_\gamma) \in H$ with at most one coordinate non-vanishing, as desired. Now apply Lemma 1. \square

Lemma 3. *The intersection of a family of subsets of a Banach space E , each of which is the locus of zeros of a continuous polynomial map, is the locus of zeros of an entire map on E .*

Proof. In view of Lemma 1, grouping together polynomials of the same degree, one can assume the family of polynomials to be countable, with $p_n: E \rightarrow F_n$ and $\deg p_1 < \deg p_2 < \dots$. One can also assume, multiplying p_n by a suitable scalar if necessary, that for every n and every $i = 0, 1, \dots, \deg p_n$ one has $\sup\{\|p_n^{(i)}(x)\|: \|x\| \leq i\} \leq 2^{-i}$. The diagonal product $p^{(i)} = \Delta_n p_n^{(i)}: E \rightarrow \oplus_n^{l_\infty} F_n$ is a homogeneous polynomial map with $\sup\{\|p^{(i)}(x)\|: \|x\| \leq i\} \leq 2^{-i}$, and the series $\sum_{i=0}^\infty p^{(i)}$ converges uniformly on every bounded set in E to the mapping $p = \Delta p_n: E \rightarrow \oplus_n^{l_\infty} F_n$. Therefore, p is an entire map ([6], Sect. 8), and also clearly $V(p) = \cap_n V(p_n)$. \square

Remark 2. An analysis of the Example in [15], p. 84 shows that the intersection of an infinite family of algebraic sets is not in general an algebraic set.

Lemma 4 (cf. [15], prop. II.1.1.1). *The union of finitely many subsets of a Banach space, each of which is the locus of zeros of a continuous polynomial map, is the locus of zeros of a continuous polynomial map.*

Proof. If p_i , $i = 1, \dots, n$ are continuous polynomial maps $E \rightarrow F_i$, then $\cap_{i=1}^n V(f_i) = V(f)$, where f is the composition of the diagonal product $\Delta f_i: x \mapsto (f_1(x), f_2(x), \dots, f_n(x)) \in F_1 \times F_2 \times \dots \times F_n$ and the canonical n -linear mapping $F_1 \times F_2 \times \dots \times F_n \rightarrow F_1 \hat{\otimes} F_2 \hat{\otimes} \dots \hat{\otimes} F_n$. \square

Lemma 5. *Let \mathcal{H}_α be a family of Hilbert spaces, let $k \in \mathbb{N}$, and let, for each α , X_α be the locus of zeros of a continuous Banach space valued k -polynomial on \mathcal{H}_α such that $0 \in X_\alpha$. The union $\cup_\alpha X_\alpha$ of the sets X_α canonically embedded into the l_2 -sum $\mathcal{H} = \oplus_\alpha^{l_2} \mathcal{H}_\alpha$ is the locus of zeros of a continuous Banach space valued $\max\{k, 2\}$ -polynomial.*

Proof. Let $X_\alpha = V(p_\alpha)$, where $p_\alpha: \mathcal{H}_\alpha \rightarrow E_\alpha$ are continuous k -polynomials. Then $\cup_\alpha X_\alpha$ is the intersection of $\cup_\alpha \mathcal{H}_\alpha$ and the intersection of the family of loci of zeros of homogeneous k -polynomials of the form $p_\alpha \circ \pi_\alpha$, where $\pi_\alpha: \mathcal{H} \rightarrow \mathcal{H}_\alpha$ is the projection. Now it remains to apply Lemmas 2 and 1. \square

The symbol $J(\mathfrak{m})$ denotes the “metrizable hedgehog of spininess \mathfrak{m} ,” that is, a bouquet of \mathfrak{m} copies of the unit interval I with a common basepoint 0_I , endowed with the maximal metric inducing the usual distance on each copy of I . ([10], Ch. 4; [3], ch. 2, problem 251).

Lemma 6. *Let \mathfrak{m} be a cardinal. The Hilbert space \mathcal{H} of weight \mathfrak{m} contains a bounded subset homeomorphic to $J(\mathfrak{m})$ in such a way that $J(\mathfrak{m})$ itself, and every element of a certain basis of closed subsets of this space, are the loci of zeros of suitable Banach space-valued continuous 2-polynomials on \mathcal{H} .*

Proof. Let I_α , $\alpha < \mathfrak{m}$, be copies of the unit interval embedded in a separable Hilbert space \mathcal{H}_α as in Assertion 2. According to Lemma 5, the union $\cup_{\alpha < \mathfrak{m}} I_\alpha$ formed in a Hilbert space $\mathcal{H} = \oplus_{\alpha < \mathfrak{m}}^{l_2} \mathcal{H}_\alpha$ is the locus of zeros of a Banach space-valued continuous 2-polynomial. It is easy to verify that $\cup_{\alpha < \mathfrak{m}} I_\alpha$ is bounded and homeomorphic to the metric hedgehog. If a closed subset $F \subseteq J(\mathfrak{m})$ contains zero, then it can be represented as the union of closed subsets $F_\alpha \subseteq I_\alpha$ each of which contains zero, and by force of Lemma 5 F is the locus of zeros of a continuous 2-polynomial map. It suffices now to check the required property for a closed subset of the form $\cup_\alpha F_\alpha$ where each F_α is a copy of the same closed subinterval $[a, 1] \subset I_\alpha$, $a > 0$. Let $f: \mathcal{H} \rightarrow \mathbb{C}$ be a linear bounded functional with $f^{-1}(\{1\}) = [a, 1] \cap I$ (Assertion 2). The operator $f^{\mathfrak{m}}: \oplus^{l_2} \mathcal{H} \rightarrow l_\infty(\mathfrak{m})$ of the form $(x_\alpha) \mapsto (f(x_\alpha))$ is bounded linear, and F is the intersection of $J(\mathfrak{m})$ with the inverse image of the constant sequence $(1) \in l_\infty(\mathfrak{m})$ under $f^{\mathfrak{m}}$. \square

Lemma 7. *Let \mathfrak{m} be a cardinal. The Hilbert space \mathcal{H} of weight \mathfrak{m} contains a subset homeomorphic to the countable Tychonoff power $J(\mathfrak{m})^{\aleph_0}$ in such a way that $J(\mathfrak{m})^{\aleph_0}$ itself, and every element of a certain basis of closed subsets of this space, are the loci of zeros of suitable Banach space-valued continuous polynomials on \mathcal{H} .*

Proof. Let $i: J(\mathfrak{m}) \rightarrow \mathcal{H}$ be a homeomorphic embedding from Lemma 6. A map from $J(\mathfrak{m})^{\aleph_0}$ into the l_2 -sum of countably many copies of \mathcal{H} , given by the rule $(x_n) \mapsto (2^{-n}i(x_n))$, is a homeomorphic embedding by force of a simple argument based upon the boundedness of $i(J(\mathfrak{m}))$ and identical to the familiar proof of the

fact that the Hilbert cube, Q^∞ (see e.g. [3], 3.4), is homeomorphic to I^{\aleph_0} . Applying Lemma 1 to the compositions of the projections $\oplus_n^{l_2} \mathcal{H}_n \rightarrow \mathcal{H}_n$ with the 2-polynomial maps determining subsets $2^{-n}i(J(\mathfrak{m}))$ in the factors $\mathcal{H}_n \cong \mathcal{H}$, one concludes that $J(\mathfrak{m})^{\aleph_0}$ the locus of zeros of a continuous 2-polynomial Banach space valued map. In view of Lemma 6, the complement to each element of a standard open subbase for $J(\mathfrak{m})^{\aleph_0}$ (a cylinder over a basic open subset of $J(\mathfrak{m})$) is the locus of zeros of a continuous 2-polynomial. Finally, the complement to each standard basic open subset in $J(\mathfrak{m})^{\aleph_0}$, being a finite union of complements to standard subbasic elements, forms by force of Lemma 4 the locus of zeros of a continuous polynomial map. \square

§4. Proof of the Main Theorem. It follows from Lemma 7 and Lemma 3 that every closed subset of $J(\mathfrak{m})^{\aleph_0}$ is homeomorphic to the locus of zeros of a suitable Banach space valued entire map on a Hilbert space \mathcal{H} of weight \mathfrak{m} : indeed, every such subset is an intersection of a family of loci of zeros of continuous polynomial maps on \mathcal{H} . But every completely metrizable space of weight $\leq \mathfrak{m}$ is homeomorphic to a closed subspace of $J = J(\mathfrak{m})^{\aleph_0}$ ([10], Problem 4.4.B.) \square

Proof of Corollary 1. A paracompact space, locally metrizable with a complete metric, is itself metrizable with a complete metric [2]. \square

Proof of Corollary 2. The Main Theorem enables one to represent any separable completely metrizable space as the locus of zeros of an entire map from a separable complex Hilbert space to a separable complex Banach space, E . It remains to compose this map with an isometric embedding of E into $C[0, 1]$ (see e.g. [12], Th. 24) and a canonical linear contraction $C[0, 1] \rightarrow \mathcal{L}^2(0, 1)$. \square

Proof of Corollary 3. A map from Corollary 2, viewed as an entire real analytic map between two real Hilbert spaces, should be composed with the norm on the image Hilbert space, which in the real case is a continuous 2-polynomial functional. \square

§5. Questions. 1. Is every complete metric space homeomorphic to the locus of zeros of an analytic map between two Hilbert spaces?

2. Is every complete metric space homeomorphic to an *algebraic* subset of a Hilbert space?

3. Let E be an infinite-dimensional Banach space of weight \mathfrak{m} . Is every complete metric space of weight $\leq \mathfrak{m}$ homeomorphic to an analytic subset of E ?

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DEPARTMENT OF MATHEMATICS, VICTORIA UNIVERSITY OF WELLINGTON, P.O. Box 600,
WELLINGTON, NEW ZEALAND

E-mail address: vladimir.pestov@vuw.ac.nz